MTH868 Qualifying Exam - version 1

Comment: 'Manifold' always means manifold without boundary.

Short answer problems: Submit three of the four problems

1. (Integration by parts) Assume M is a compact oriented manifold of dimension $n+1$ with boundary $\partial M \neq \emptyset$ carrying the induced orientation from M. If ω is a p-form and τ is an $(n-p)$-form prove that

$$\int_M d\omega \wedge \tau = \int_{\partial M} \omega \wedge \tau + (-1)^{p+1} \int_M \omega \wedge d\tau.$$

2. Is it true that $\tau \wedge \tau = 0$ for any differential form τ on $\mathbb{R}^n$? Explain why or why not.

3. Give an example of an orientation form on the n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Explain why it is an orientation form.

4. Let X, Y be manifolds, where X is k-dimensional, compact and oriented. Assume further that $f_0, f_1 : X \to Y$ are homotopic smooth maps. If $\omega \in \mathcal{A}^k(Y)$ is a closed form prove that

$$\int_X f_0^* \omega = \int_X f_1^* \omega.$$

Solve four of the following five problems:

5. Assume $f : S^n \to S^n$ is a smooth map of degree different from $(-1)^{n+1}$. Show that f must have a fixed point, i.e. a point x for which $f(x) = x$.

6. Assume $m < n$, M is an m-dimensional manifold, and $\phi : M \to S^n$ is a smooth map. Show that ϕ is homotopy equivalent to a constant map.

7. Show that the set

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid (4x^2(1-x^2) - y^2)^2 + z^2 = \frac{1}{4}\}$$

is a two dimensional submanifold of $\mathbb{R}^3$.

8. Use the Meyer-Vietoris sequence to compute $H^k(S^n \times S^m)$ where $n, m \geq 1$, $k \geq 0$.

9. Let G be a finite group acting on a manifold M so that M/G becomes a manifold. Let $\pi : M \to M/G$ be the projection. Show that the map $\pi^* : H^p(M/G) \to H^p(M)$ is injective.

Solve the following problem

10. Assume $M, \tilde{M}$ are compact manifolds of the same dimension, and let $\pi : \tilde{M} \to M$ be a d-fold covering, i.e. π is a smooth map, and M can be covered with open sets U such that $\pi^{-1}(U)$ is a disjoint union of open sets $U_1, \ldots, U_d \subset \tilde{M}$ so that the maps $\pi|_{U_j} : U_j \to U$ are diffeomorphisms for all $1 \leq j \leq d$.

   (a) Prove that $\chi(\tilde{M}) = d\,\chi(M)$.

   (b) Use the fact that the quotient map $\pi : S^n \to \mathbb{R}P^n = S^n/\mathbb{Z}_2$ is a 2-fold covering to find $\chi(\mathbb{R}P^n)$.

   (c) Using the statement of problem #9 compute $H^p(\mathbb{R}P^n)$ for $0 \leq p \leq n$.